A SCHRÖDINGER FORMULATION OF BIANCHI I SCALAR FIELD COSMOLOGY

Jennie D'Ambroise
Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003, USA
e-mail: dambroise@math.umass.edu

Abstract: We show that the Bianchi I Einstein field equations in a perfect fluid scalar field cosmology are equivalent to a linear Schrödinger equation. This is achieved through a special case of the recent FLRW Schrödinger-type formulation, and provides an alternate method of obtaining exact solutions of the Bianchi I equations.
AMS Subj. Classification: 83C05, 83C15
Key Words: Einstein field equations, Bianchi I universe, Friedmann-Lemaître-Robertson-Walker universe, perfect fluid, Schrödinger equation.

1. Introduction

Recently, a correspondence was established between solutions of Einstein's field equations in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe and solutions of a particular nonlinear Schrödinger-type differential equation. That is, given a solution of the latter, a solution of the former can be constructed via the prescription given in [3] (and vice versa). Further motivated by a connection between Bianchi I and FLRW cosmologies seen in a paper by James E. Lidsey [6], an analogous <u>linear</u> Schrödinger formulation is demonstrated here for the anisotropic Bianchi I universe. The author would like to extend many thanks to Floyd L. Williams for his valued advice on the results presented here.

2. Einstein Equations

Consider the Einstein field equations $T_{ij} = K^2 G_{ij}$ for a perfect fluid Bianchi I universe with scalar field ϕ, potential V and metric $ds^2 = -dt^2 + X(t)^2dx^2 + Y(t)^2dy^2 + Z(t)^2dz^2$. We will consider the case where the energy density and pressure are given solely by a scalar field contribution, i.e. no matter contribution. That is, $\rho = \dot{\phi}^2/2 + V \circ \phi$ and $p = \dot{\phi}^2/2 - V \circ \phi$. For a vanishing cosmological constant the equations take the form

$$\begin{aligned}
\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{X}\dot{Z}}{XZ} + \frac{\dot{Y}\dot{Z}}{YZ} &\overset{(i)}{=} K^2\rho \\
\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} &\overset{(ii)}{=} -K^2 p \\
\frac{\ddot{X}}{X} + \frac{\ddot{Z}}{Z} + \frac{\dot{X}\dot{Z}}{XZ} &\overset{(iii)}{=} -K^2 p \\
\frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\dot{Y}\dot{Z}}{YZ} &\overset{(iv)}{=} -K^2 p
\end{aligned} \tag{2.1}$$

where $K^2 = 8\pi G$ and G is Newton's constant.

The fluid conservation equation can be derived from these equations and is

$$\dot{\rho} + \theta(\rho + p) = 0$$

where

$$\theta \equiv \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) \tag{2.2}$$

is the expansion/contraction of volume. By definitions of ρ and p, this equation reduces to the Klein-Gordon equation of motion

$$\ddot{\phi} + \theta\dot{\phi} + V' \circ \phi = 0.$$

Note that for $\gamma = 2\dot{\phi}^2/[\dot{\phi}^2 + 2(V \circ \phi)]$ one has the equation of state $p = (\gamma - 1)\rho$.

3. Description of The Correspondence $(X, Y, Z, \phi, V) \longleftrightarrow u$

Similar to the formulation in Lidsey [6], we first describe how a solution to the Bianchi I equations (i)-(iv) can be used to derive a solution to the FLRW equations. Since [3] provides the FLRW-Schrödinger connection, this will motivate the Schrödinger-Bianchi I correspondence.

We begin by defining the quantities

$$\eta_1 \equiv \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}, \quad \eta_2 \equiv \frac{\dot{X}}{X} - \frac{\dot{Z}}{Z}, \quad \eta_3 \equiv \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z}. \tag{3.1}$$

Computing $\frac{1}{2}[(i) - (ii) - (iii) - (iv)]$ and using the definitions above, one can verify Raychaudhuri's equation

$$\dot{\theta} + 2\mu^2 - \frac{1}{9}\theta^2 + \frac{K^2}{2}(3p + \rho) = 0 \tag{3.2}$$

where μ is the shear scalar given by $\mu^2 \equiv \frac{1}{6}\left(\eta_1^2 + \eta_2^2 + \eta_3^3\right) + \frac{2}{9}\theta^2$. Also, rewriting equation (i) using the above notation,

$$\frac{5}{9}\theta^2 - \mu^2 = K^2\rho. \tag{3.3}$$

Remarkably, equations (3.2) and (3.3) are a special case of the FLRW field equations (see [3]) with the following substitutions:

$$n = 6, \quad k = 0, \quad a(t) = (XYZ)^{1/3}, \quad D = \frac{X^2Y^2Z^2}{6K^2}\left(\eta_1^2 + \eta_2^2 + \eta_3^2\right). \tag{3.4}$$

Note that [3] requires D to be constant. By (i)-(iv) in (2.1) and the following lemma, one can easily show that each of the products $XYZ\eta_i$ for $i = 1, 2, 3$ is a constant function of t.

Lemma. *For arbitrary functions $X(t), Y(t), Z(t) > 0$ and $f(t)$,*

$$\dot{f} + \theta f = 0 \Longleftrightarrow fXYZ \text{ is a constant function}$$

for θ as in (2.2) in terms of X, Y, Z.

Therefore, given a quintet (X, Y, Z, ϕ, V) solution to (i)-(iv) in (2.1), one can construct a solution to the FLRW field equations by (3.4). By the converse of Theorem 1 from [3], one can then construct a solution to the time-independent linear Schrödinger equation

$$u''(x) + [E - P(x)]u(x) = 0. \tag{3.5}$$

with constant energy E and potential $P(x)$. We will see that the Schrödinger solutions derived in this way will always be such that $E < 0$.

In this paper, we show the direct correspondence $(X, Y, Z, \phi, V) \longleftrightarrow u$ between solutions (X, Y, Z, ϕ, V) of (i)-(iv) in (2.1) and solutions u of (3.5). This correspondence provides an alternate method of solving Bianchi I field equations.

With the above notation in place, we can now state the main theorem:

Theorem. *Let $u(x)$ be a solution of equation (3.5), given $E < 0$ and $P(x)$. Then a solution (X, Y, Z, ϕ, V) of the Einstein equations (i)-(iv) in (2.1) can be constructed as follows. First choose functions $\sigma(t)$, $\psi(x)$ such that*

$$\dot{\sigma}(t) = u(\sigma(t))\,, \quad \psi'(x)^2 = \frac{2}{3K^2}P(x) \tag{3.6}$$

and also constants c_1, c_2 such that

$$c_1^2 + c_1c_2 + c_2^2 = -\frac{4E}{3}. \tag{3.7}$$

Next define the functions

$$R(t) = u(\sigma(t))^{-1/3} \tag{3.8}$$

and

$$\alpha(t) = \frac{c_1}{2}\sigma(t), \quad \beta(t) = \frac{c_2}{2}\sigma(t), \quad \gamma(t) = -\alpha(t) - \beta(t). \tag{3.9}$$

Then the following quintet solves Einstein's field equations (i)-(iv):

$$X(t) = R(t)e^{\alpha(t)}, \quad Y(t) = R(t)e^{\beta(t)}, \quad Z(t) = R(t)e^{\gamma(t)}, \tag{3.10}$$

$$\phi(t) = \psi(\sigma(t)), \quad V = \frac{1}{3K^2}\left[(u')^2 + u^2[E - P]\right] \circ \psi^{-1}. \tag{3.11}$$

Here, in fact, (X, Y, Z, ϕ, V) will also satisfy the equations

$$\begin{aligned} \dot{\phi}^2 &= \frac{2}{3K^2}\left(-\dot{\theta} + \frac{E}{X^2Y^2Z^2}\right) && (3.12)\\ V(\phi(t)) &= \frac{1}{3K^2}\left(\theta^2 + \dot{\theta}\right). && (3.13) \end{aligned}$$

Conversely, let (X, Y, Z, ϕ, V) be a solution of equations (i)-(iv) in (2.1), with ρ and p as before. Similar to (3.6), choose some solution $\sigma(t)$ of the equation

$$\dot{\sigma}(t) = \frac{1}{XYZ}. \tag{3.14}$$

Then equation (3.5) is satisfied for

$$\begin{aligned} E &= -\frac{1}{2}X^2Y^2Z^2\left(\eta_1^2 + \eta_2^2 + \eta_3^2\right), && (3.15)\\ P(x) &= \frac{3}{2}K^2\left[\dot{\phi}^2X^2Y^2Z^2\right] \circ \sigma^{-1}(x), \\ u(x) &= \left[\frac{1}{XYZ}\right] \circ \sigma^{-1}(x). \end{aligned}$$

Note that in (3.15), $E < 0$ and is constant by the same argument stated for D above the lemma. The theorem therefore provides a concrete correspondence $(X, Y, Z, \phi, V) \leftrightarrow u$ between solutions (X, Y, Z, ϕ, V) of the field equations (i)-(iv) and solutions u of the linear Schrödinger equation (3.5).

Remarks.

1. The case $P(x) = 0$:

   In most examples $P(x)$ is nonzero. However, if $P(x) = 0$ then the theorem must be stated carefully, as $\psi(x)$ is a constant function by (3.6) and has no inverse. Therefore the expression for V in (3.11) has no meaning. In this case, we will show that the right-hand side of (3.13) is a constant function that will serve as our new definition for V. By (3.8)-(3.10), $u \circ \sigma = 1/(XYZ)$. Differentiating and using (3.6), $(u' \circ \sigma)(u \circ \sigma) = -\theta/(XYZ)$. That is, $u' \circ \sigma = -\theta$. Differentiating again, $(u'' \circ \sigma)(u \circ \sigma) = -\dot{\theta}$. Using these in (3.5), composed with σ and multiplied by $u \circ \sigma$,

   $$-\dot{\theta} + \frac{E}{X^2Y^2Z^2} = 0. \tag{3.16}$$

   Therefore (3.12) is still valid, of course, with the left side equal to zero since ϕ is constant in this case by (3.11). Differentiating (3.16),

   $$-\ddot{\theta} - \frac{2E\theta}{X^2Y^2Z^2} = 0. \tag{3.17}$$

   Now, to show that (3.13) is constant, differentiate its right side to get

   $$\begin{aligned} \frac{d}{dt}\left\{\theta^2 + \dot{\theta}\right\} &= 2\theta\dot{\theta} + \ddot{\theta} \\ &= 2\theta\left(\dot{\theta} - \frac{E}{X^2Y^2Z^2}\right) \text{ by (3.17)} \\ &= 0 \text{ by (3.16)}. \end{aligned}$$

   Therefore in the case $P(x) = 0$ when the definition of V in (3.11) no longer has meaning, we define $V(x) = V_0 \equiv (\theta^2 + \dot{\theta})/(3K^2)$ and $\phi(t) =$ any constant; and we note that equations (3.12) and (3.13) still hold in this special case.

2. Equations (3.12) and (3.13) imply (i)-(iv) in (2.1) under a condition:

   By the comment preceding the lemma above, any solution to the equations (i)-(iv) will have the property that $XYZ\eta_i$ is constant for η_i as in (3.1) and $i = 1, 2, 3$. Suppose we are given a priori (positive) functions X, Y, Z with this property. Differentiating each of $XYZ\eta_i$ and setting equal to zero shows exactly that the left sides of (ii)-(iv) are equal to each other. Next, any three positive functions can be reparametrized as in (3.10) for $R(t) = (XYZ)^{1/3}$, $\alpha = \frac{1}{3}\ln\left(\frac{X^2}{YZ}\right)$, $\beta = \frac{1}{3}\ln\left(\frac{Y^2}{XZ}\right)$ and $\gamma = \frac{1}{3}\ln\left(\frac{Z^2}{XY}\right)$. Using these formulas and the constant quantities $XYZ\eta_i$, one can easily compute that each of $\dot{\alpha}, \dot{\beta}$ and $\dot{\gamma}$ are scalar multiples of $1/XYZ$, therefore establishing (3.9) for σ as in (3.14). Finally, under the condition that c_1, c_2 satisfy (3.7) for the constant $E < 0$ given by X, Y, Z as in (3.15), one can show that equations (3.12) and (3.13) indeed compute ϕ and V solving (i)-(iv) in terms of the given X, Y, Z.

4. Examples

As an illustration of the theorem, take the solution $u(x) = Ae^{-\sqrt{-E}x} - Be^{\sqrt{-E}x}$ for $A, B > 0$ to the equation (3.5) with $E < 0$ and $P(x) = 0$. Solving the differential equation (3.6) for σ using Mathematica, we obtain

$$\sigma(t) = \frac{1}{2\sqrt{-E}} \ln\left[\frac{A}{B}\tanh^2[\sqrt{-ABE}(t-c_0)]\right] \tag{4.18}$$

for integration constant c_0. Also by (3.6), $\psi(x) = \psi_0 \equiv$ any constant. Then by (3.8)

$$R(t) = u \circ \sigma^{-1/3} = \left(\frac{1}{2\sqrt{AB}}\sinh[2\sqrt{-ABE}(t-c_0)]\right)^{1/3}. \tag{4.19}$$

Further let c_1, c_2 be *any* constants such that (3.7) holds given the constant choice E. We form X, Y, Z, ϕ according to (3.9)-(3.11) and obtain

$$\begin{aligned}
X &= \left(\frac{1}{2\sqrt{AB}}\sinh[2\sqrt{-ABE}(t-c_0)]\right)^{1/3}\left(\sqrt{\frac{A}{B}}\tanh[\sqrt{-ABE}(t-c_0)]\right)^{c_1/(2\sqrt{-E})} \qquad (4.20)\\
Y &= \left(\frac{1}{2\sqrt{AB}}\sinh[2\sqrt{-ABE}(t-c_0)]\right)^{1/3}\left(\sqrt{\frac{A}{B}}\tanh[\sqrt{-ABE}(t-c_0)]\right)^{c_2/(2\sqrt{-E})}\\
Z &= \left(\frac{1}{2\sqrt{AB}}\sinh[2\sqrt{-ABE}(t-c_0)]\right)^{1/3}\left(\sqrt{\frac{A}{B}}\tanh[\sqrt{-ABE}(t-c_0)]\right)^{-(c_1+c_2)/(2\sqrt{-E})}
\end{aligned}$$

for $t > c_0$ and $\phi = \psi_0$. Since $P = 0$, ψ^{-1} does not exist (see Remark 1) and we use (3.13) for the definition of V and obtain the constant $V = V_0 \equiv -4ABE/(3K^2)$. The reader may compare this solution with a similar one in [1].

As another example, we will begin with the same assumptions on E, P and will obtain quite a different Einstein solution. That is, again let $E < 0$ and $P(x) = 0$, but take solution $u(x) = Ae^{-\sqrt{-E}x}$ to (3.5) with $A > 0$. Solving the differential equations in (3.6), we obtain $\sigma(t) = \ln\left[A\sqrt{-E}(t-c_0)\right]/\sqrt{-E}$ for $t > c_0$ and $\psi(x) = \psi_0 \equiv$ any constant (therefore we will also have $\phi = \psi_0$ by (3.11)). By (3.8), $R(t) = (\sqrt{-E}(t-c_0))^{1/3}$. Letting c_1, c_2 be any solution to (3.7), finally we compute X, Y, Z to be

$$\begin{aligned}
X &= A^{c_1/(2\sqrt{-E})}\left(\sqrt{-E}(t-c_0)\right)^{c_1/(2\sqrt{-E})+1/3} \qquad (4.21)\\
Y &= A^{c_2/(2\sqrt{-E})}\left(\sqrt{-E}(t-c_0)\right)^{c_2/(2\sqrt{-E})+1/3}\\
Z &= A^{-(c_1+c_2)/(2\sqrt{-E})}\left(\sqrt{-E}(t-c_0)\right)^{-(c_1+c_2)/(2\sqrt{-E})+1/3}
\end{aligned}$$

by (3.9)-(3.10) for $t > c_0$. Again, since $P = 0$, ψ^{-1} does not exist and we use (3.13) as the definition for V and obtain $V = 0$. That is, this solution is vacuum.

To further demonstrate the utility of the theorem, we will take a trivial non-physical solution of (i)-(iv), and first use the converse theorem to map to a solution of (3.5). We will then apply the theorem a second time and map back to a solution of (i)-(iv) and will have produced a physically acceptable example. Begin by considering the vacuum ($\phi = V = 0$) solution

$$X = R_0 e^{\alpha_0}, \quad Y = R_0 e^{\beta_0}, \quad Z = R_0 e^{\gamma_0} \tag{4.22}$$

for constants $R_0 > 0, \alpha_0, \beta_0, \gamma_0$ such that $\alpha_0 + \beta_0 + \gamma_0 = 0$. By (3.1) and (3.15), $E = P = 0$ and $u(x) = u_0 \equiv (1/R_0^3)$. Clearly this is a solution to the linear Schrödinger equation (3.5). Note that since u is constant and ϕ is zero, we did not need to compute σ. Now to map back to a solution of (i)-(iv), we solve (3.6) and use (3.11) so that $\sigma(t) = (1/R_0^3)t$ and $\psi = \phi =$ any constant. Now by (3.8)-(3.10),

$$X = R_0 e^{c_1 t/(2R_0^3)}, \quad Y = R_0 e^{c_2 t/(2R_0^3)}, \quad Z = R_0 e^{-(c_1+c_2)t/(2R_0^3)}. \tag{4.23}$$

Again by (3.13), $V = 0$.

As a final example, we take $u(x) = (1/x)e^{Ex^2/2}$, $E < 0$ and $P(x) = (2/x^2) + E^2x^2$ for $x > 0$. This Schrödinger solution was found using the techniques in [5]. Solving (3.6), we obtain $\sigma(t) = \sqrt{-\frac{2}{E}\ln[-E(t-c_0)]}$ for integration constant c_0 and

$$\psi(x) = \frac{1}{\sqrt{6}K}\left(\sqrt{2+E^2x^4} + \sqrt{2}\ln\left[\frac{x^2}{2+\sqrt{4+2E^2x^4}}\right]\right). \tag{4.24}$$

Graphing this function for a few values of E indicates that the inverse exists, and we will denote it by ψ^{-1}. Calculating X, Y, Z, ϕ, V according to the theorem,

$$\begin{aligned} X &= \left((t-c_0)\sqrt{-2E\ln[-E(t-c_0)]}\right)^{1/3} e^{c_1/\sqrt{\ln[-E(t-c_0)]/(-2E)}} \qquad (4.25)\\ Y &= \left((t-c_0)\sqrt{-2E\ln[-E(t-c_0)]}\right)^{1/3} e^{c_2/\sqrt{\ln[-E(t-c_0)]/(-2E)}} \\ Z &= \left((t-c_0)\sqrt{-2E\ln[-E(t-c_0)]}\right)^{1/3} e^{-(c_1+c_2)/\sqrt{\ln[-E(t-c_0)]/(-2E)}} \\ V &= \frac{1}{3K^2}\left[-\frac{e^{Ex^2}}{x^4}\left(1+Ex^2\right)\right] \circ \psi^{-1}(t) \end{aligned}$$

$$\phi = \frac{1}{\sqrt{3}K}\left(\sqrt{1+2\ln^2[-E(t-c_0)]} + \ln\left(\frac{-\ln[-E(t-c_0)]}{E+E\sqrt{1+2\ln^2[-E(t-c_0)]}}\right)\right)$$

for c_1, c_2 satisfying (3.7) and $t > c_0$.

References

[1] R. Bali, C. Jain, Bianchi I inflationary universe in general relativity, *Pramana - Journal of Physics*, **59** (2002), No. 1, 1-7.

[2] Dadhich, Naresh, Derivation of the Raychaudhuri Equation, (2005), arXiv: gr-gc/0511123 v2.

[3] J. DAmbroise, F.L. Williams, A Non-linear Schrödinger Type Formulation of FLRW Scalar Field Cosmology, *International Journal of Pure and Applied Mathematics*, **34** (2006), No. 1, 117.

[4] R. Hawkins, J. Lidsey, The Ermakov-Pinney equation in scalar field cosmologies, *Physical Review*, **D66** (2002), 023523-023531.

[5] G. Levai, A search for shape-invariant solvable potentials, *Journal of Physics*, **A22** (1989), 689-702.

[6] J. Lidsey, Multiple and anisotropic inflation with exponential potentials, *Class. Quantum Grav.*, **9** (1992), 1239-1253.